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The interaction trapping of internal gravity waves

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It is shown that as a result of their non-linear interactions, internal gravity waves in an unbounded fluid can be trapped to a layer of finite depth by periodic small variations in either the density gradient or in a weak horizontal steady current. This trapping occurs when the vertical component of the wave-number is half that of the density gradient or of the current variations. The energy density of the wave motion trapped near the ocean surface decreases exponentially with depth over a distance that is inversely proportional to the magnitude of the variations in density gradient or in horizontal current speed.

1. Introduction

The atmosphere and the oceans are almost always stably stratified. The degree of stratification, described by the Brunt–Väisälä or buoyancy frequency

$$N = \left\{-\frac{g}{\rho_0}\frac{\partial\bar{\rho}}{\partial y}\right\}^{\frac{1}{2}},$$

usually varies in the vertical, often in an irregular way. Moreover, in the absence of topographical effects, the quasi-steady motion usually consists of steady horizontal streaming at speeds that may vary with height or depth. It is of interest to examine the influence that these variations have on the propagation of other internal wave disturbances in the medium.

An account of the simple properties of small-scale internal waves has been given by Phillips (1966). If the buoyancy frequency N is constant in the region of interest, the frequency n of internal wave modes is specified by the inclination θ of the wave-number to the horizontal:

$$n = N\cos\theta.$$

When $\theta \to \frac{1}{2}\pi$, $n \to 0$, so that the steady horizontally uniform disturbances of the ocean or atmosphere might in certain circumstances be regarded as the limit of zero frequency internal waves. The interaction between this motion and other internal waves could then be considered as a special (but rather interesting) case of the more general mutual interaction of internal waves.

For the weak second-order interactions among waves, with wave-numbers $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ and frequencies n_1, n_2 and n_3, the resonance conditions

$$\mathbf{k}_1 - \mathbf{k}_2 = \mathbf{k}_3,$$

$$n_1 - n_2 = n_3,$$

must be satisfied simultaneously. In this situation, if $\mathbf{k}_3$ is taken vertically, $n_3 = 0$; if resonant interactions are to occur $\theta_1 = \theta_2$ and the wave-numbers form an isosceles triangle (figure 1). At the Sixth Symposium in Naval Hydrodynamics in 1966 at Washington, D.C., I presented a simple analysis of the problem from this point of view and some interesting and rather unexpected results emerged.

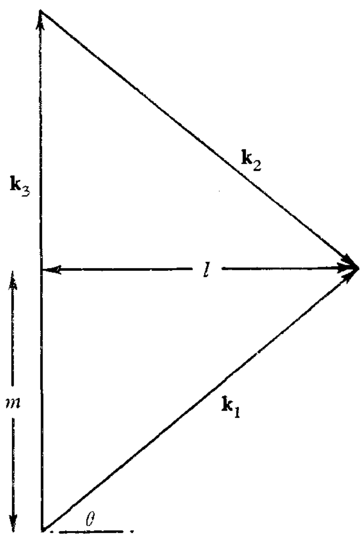

FIGURE 1

The interactions were studied as an initial value problem: it was supposed that at time $t = 0$, the motion consisted of the steady streaming superimposed on a uniform wave with wave-number $\mathbf{k}_1$, the amplitude of the third component being initially zero. It was found that the energy of the $\mathbf{k}_1$ component only is transferred to the wave-number $\mathbf{k}_2$; that the steady streaming motion acts as a catalyst, playing an essential part in the interaction but not itself entering into the energy exchange. If only this triplet of wave-numbers is present, the energy flows back and forth between the two inclined wave-numbers as a result of their interaction with the third (vertical) wave-number. The net energy flux in physical space is then alternately upwards and downwards and this result suggested the possibility of trapping; that an irregular streaming motion might limit the vertical extent that internal gravity waves could propagate from their point of origin even if the buoyancy frequency were constant. The depth of a trapped zone was inferred by group velocity arguments; it was found to be finite for any inclination θ. The case was far from proved, however, since these simple solutions were to the initial value problem involving internal gravity wave trains that were spatially uniform; one purpose of this paper is to provide a more detailed discussion that establishes this trapping effect explicitly.

In commenting on this paper, Dr T. B. Benjamin (1968) presented an alternative analysis in which the problem was viewed from the point of view of single scattering theory; the wave being *scattered* by the spatially periodic current field. He found that the internal wave generally attenuated by scattering within a depth of the same order as that found in the previous analysis but that there was an exceptional case $\theta = \frac{1}{4}\pi$ in which the scattering cross-section vanished—the waves continued to propagate without attenuation at all. Although these were two rather different problems (the interaction problem corresponding more to multiple scattering), one would have expected the results to be in qualitative agreement, and the appearance of the 'window' in Brooke Benjamin's analysis but not in Phillips's was a matter of some concern.

It has subsequently appeared that part of the discrepancy arises from the view of the steady spatially periodic motion as the limit of a zero frequency internal wave. In an internal gravity wave the magnitude of the variations in density δ_ρ and velocity $\delta \mathbf{u}$ are intimately related

$$|\delta\rho| = \frac{|\delta \mathbf{u}|}{N}\frac{\partial \bar{\rho}}{\partial z},$$

and this relation is preserved as $n \to 0$. In the simple interaction analysis, then, there are coupled variations in both the density gradient and the horizontal velocity field, whereas in Brooke Benjamin's scattering approach only velocity variations were considered. In actual fact, of course, either or both can occur independently; it is of interest to see how the two effects separately influence the nature of the trapping. The exploration of this and the evaluation of its relevance to oceanography is the second aim of this study.

2. The interaction equations

Consider a region of stratified incompressible fluid in which, in the basic state, the buoyancy frequency N is independent of the vertical (y) co-ordinate. The x co-ordinate is horizontal and can most conveniently be taken to lie in the plane defined by the two wave-number vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. Only the x component of the steady horizontal shearing motion will be found to be relevant to the interaction; it will be supposed to vary periodically in the vertical with wave-number

$$\mathbf{k}_3 = (0,\, 2m,\, 0).$$

Thus

$$U(y) = (U \sin 2my, 0,\, W(y)). \tag{2.1}$$

Associated with this there may or may not be periodic variations in the (time independent) density gradient about its mean value, so that in general

$$\frac{\partial \rho}{\partial y} = -\frac{\rho_0 N^2}{g}\{1 + r\sin(2my+\gamma)\}, \tag{2.2}$$

where γ is an arbitrary phase angle. For local static stability the density gradient must everywhere decrease in the vertical, so that $r < 1$. The buoyancy of the fluid relative to a reference state with density ρ_0 is defined as $g(\rho_0 - \rho)/\rho_0$ so that the periodic variations b in buoyancy are

$$b = N^2 \frac{r}{2m}\cos(2my+\gamma). \tag{2.3}$$

The gradient in mean buoyancy is, from (3.2) simply N^2.

We are concerned with the interactions between this steady field and the pair of internal waves specified by figure 1. Since the z-direction was chosen to be orthogonal to all three wave-number vectors, the motion is independent of this co-ordinate, and a stream function ψ can be defined such that

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x. \tag{2.4}$$

The vorticity equation in the stratified fluid can be expressed as

$$\frac{\partial}{\partial t}\nabla^2\psi - \frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)} + \frac{\partial b}{\partial x} = 0, \tag{2.5}$$

while the continuity condition

$$\frac{\partial \rho}{\partial t} + \mathbf{u}\,.\,\nabla\rho = 0$$

can be expressed in terms of the buoyancy field as

$$\frac{\partial b}{\partial t} + \frac{\partial(b,\psi)}{\partial(x,y)} - N^2\frac{\partial\psi}{\partial x} = 0. \tag{2.6}$$

If the non-linear terms are neglected, there results the equation

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi + N^2\frac{\partial^2\psi}{\partial x^2} = 0,$$

appropriate to infinitesimal internal gravity waves. Solutions of the type

$$\left.\begin{aligned} \psi &= A\cos(lx \pm my - nt + \delta), \\ b &= -(N^2 l/n)A\cos(lx \pm my - nt + \delta) \end{aligned}\right\} \tag{2.7}$$

exist, provided $$n = \frac{l}{(l^2+m^2)^{\frac{1}{2}}}N = Nl/k = N\cos\theta,$$

where θ is the inclination of the wave-number vector to the horizontal. The presence of the non-linear terms in (2.5) and (2.6), however, results in a coupling between the two internal waves and the steady field; if the energy density of the wave motion is sufficiently small, the interactions will be weak and of a resonant kind rather than the strong indiscriminate type characteristic of turbulence in a stratified fluid (Phillips 1966). The exact criterion for this will emerge shortly; it is sufficient at this stage to anticipate that the weak interactions will result in variations in the wave amplitude (and possibly the phases) over scales that are large compared with the wavelength involved.

We seek, in particular, solutions to the non-linear equations that represent a trapping of the energy of the internal waves into a region of generally finite vertical extent, characterized by some length scale ϵ^{-1}. Consequently, let

$$\left.\begin{aligned} \psi &= (2m)^{-1}U\{\alpha\cos 2my + f_1(\epsilon y)\cos\chi_1 + g_1(\epsilon y)\cos\zeta_1 + h_1(x,y,t)\}, \\ b &= -\frac{NUk}{2m}\{\beta\cos(2my+\gamma) + f_2(\epsilon y)\cos\chi_2 + g_2(\epsilon y)\cos\zeta_2 + h_2(x,y,t)\}, \end{aligned}\right\} \tag{2.8}$$

where U is a scale for the speed of the horizontal motion, $k^2 = l^2+m^2$,

$$\left.\begin{aligned}\chi_i &= lx+my-nt+\delta_i(\epsilon y)\\ \zeta_i &= lx-my-nt+\Delta_i(\epsilon y)\end{aligned}\right\} \quad (i = 1,2), \tag{2.9}$$

and h_1, h_2 are the non-resonant products of the interaction. The dimensionless quantities α and β determine the magnitude of the steady current and the steady variations in the density gradient; the case $\alpha = 1, \beta = 2m/k$ corresponding to the zero frequency internal wave limit and $\alpha = 1$, $\beta = 0$ to the problem of Brooke Benjamin. The functions f and g specify the amplitudes of the two internal wave motions. The scale ϵ^{-1} is to be determined by the strength of the interaction; as $\epsilon \to 0$, the interaction vanishes and from (2.7) the sets of quantities f_i, g_i, δ_i and Δ_i become independent of y and equal in pairs. Moreover, with the weakly interacting waves, we would expect (2.7) to be a *local* representation of the wave motion, so that

$$\left.\begin{aligned}f_2 &= f_1[1+O(\epsilon/m)],\\ g_2 &= g_1[1+O(\epsilon/m)].\end{aligned}\right\} \tag{2.10}$$

In seeking solutions of this kind, the procedure is in some ways analogous to the two-time expansion in classical mechanics, a technique used in the past in initial value wave interaction problems. In this case, however, we are concerned with two *spatial* scales—one representative of the wavelengths of the interacting waves and the other, ϵ^{-1}, specifying the scale of depth over which the energy density of the field varies.

In order to find solutions for the wave amplitudes f_i and g_i, we must substitute the expressions (2.8) into the equations of motion (2.5) and (2.6). It is found readily that

$$\begin{aligned}\nabla^2\psi = &-2Um\alpha\cos 2my-(U/2m)(l^2+m^2)\{f_1\cos\chi_1+g_1\cos\zeta_1\}\\ &-\epsilon U\{f_1'\sin\chi_1-g_1'\sin\zeta_1+f_1\delta_1'\cos\chi_1-g_1\Delta_1'\cos\zeta_1\}\\ &+\epsilon^2(U/2m)\{f_1''\cos\chi_1+g_1''\cos\zeta_1+\ldots\}.\end{aligned} \tag{2.11}$$

Since we suppose that the scale ϵ^{-1} of the variation of wave amplitude is large compared with the vertical wavelength $(2m)^{-1}$, then $\epsilon/2m \ll 1$ and the terms of order ϵ^2 in this expression can usually be neglected. Notice, however, that these terms contain the highest order derivatives; the problem involves not a regular but a singular perturbation and we must anticipate the possibility of local regions of the motion where either the wave amplitude or the gradient of the amplitude changes sufficiently rapidly that these terms become locally comparable with the lower order term.

The substitution of (2.11) and (2.8) into the vorticity equation (2.5) leads, after some algebra, to

$$\begin{aligned}-\frac{Un}{2m}(l^2+m^2)\{f_1\sin\chi_1+g_1\cos\zeta_1\}+\frac{NUkl}{2m}\{f_2\sin\chi_2+g_2\sin\zeta_2\}\\ +\epsilon nU\{f_1'\cos\chi_1-g_1'\cos\zeta_1-f_1\delta_1'\sin\chi_1+g_1\Delta_1'\sin\zeta_1\}\\ -\frac{U^2l}{4m}(l^2-3m^2)\alpha\{f_1\cos(\zeta_1+\phi_1)-g_1\cos(\chi_1-\phi_1)\}+h = 0,\end{aligned} \tag{2.12}$$

correct to $O(\epsilon)$, where $\phi_1 = \delta_1-\Delta_1$ represents the phase difference between the two inclined wave trains and h the non-resonant interaction terms, those at wave-numbers other than $(0, 2m)$ or $(l \pm m)$. It is interesting to notice that in the non-linear terms of (2.5), the contribution involving cross-products of χ_1 and ζ_1 terms vanishes identically; no term arises in this equation to give interaction with the vertical wave-number $2m$.

If, further, in the light of (2.10), we let

$$f_2 = f_1(\epsilon y) + (\epsilon/m)\hat{f}(\epsilon y), \tag{2.13}$$

with similar expressions for g_2, δ_2 and Δ_2 defining $\hat{g}$, $\hat{\delta}$ and $\hat{\Delta}$ respectively and substitute these into (2.12), the resulting equation has terms, each of which contains a factor sine or cosine of either χ_1 or ζ_1. These groups of terms can be separated by multiplying in turn by these factors and averaging locally and there result the following four equations:

$$\epsilon n f_1' + \frac{\epsilon N k l}{2m^2} f_1 \hat{\delta} + \left\{\frac{Ul}{4m}(l^2 - 3m^2)\alpha\cos\phi_1\right\} g_1 = 0, \tag{2.14}$$

$$-\epsilon n g_1' + \frac{\epsilon N k l}{2m^2} g_1 \hat{\Delta} - \left\{\frac{Ul}{4m}(l^2 - 3m^2)\alpha\cos\phi_1\right\} f_1 = 0, \tag{2.15}$$

$$-\epsilon n f_1 \delta_1' + \frac{\epsilon N k l}{2m^2}\hat{f} + \left\{\frac{Ul}{4m}(l^2 - 3m^2)\alpha\sin\phi_1\right\} g_1 = 0, \tag{2.16}$$

$$\epsilon n g_1 \Delta_1' + \frac{\epsilon N k l}{2m^2}\hat{g} + \left\{\frac{Ul}{4m}(l^2 - 3m^2)\alpha\sin\phi_1\right\} f_1 = 0, \tag{2.17}$$

where the wave frequency $n = Nl/k$.

A similar series of substitutions into the buoyancy equation can be made. Again it is found that the non-linear terms generate no contribution to the vertical wave-number $2m$. This fact is significant; it implies that the horizontal streaming motion and the variations in density gradient are not themselves affected by the interaction; they are catalytic in the sense that they result in an energy exchange between the other two wave modes but they do not themselves partake in it. Such a characteristic was found in the initial value, time-dependent solutions (Phillips 1968). In this respect the particular type of interaction described here is probably better considered as a multiple scattering process in contrast with the more general interaction situation in which all wave-numbers participate in the energy exchanges. It might be noted that this situation represents an exception to the result given by Hasselmann (1967) that a wave motion with any wave-number is unstable to disturbances that form with it a resonant triad. The algebra involved in the substitution of (2.8) into the buoyancy equation (2.6) is straightforward and leads to a further set of four equations:

$$\{(\alpha - \beta\cos\gamma)\cos\phi_1 - \beta\sin\gamma\sin\phi_1\} f_1 + \frac{2\epsilon N}{mkU} g_1 \hat{\Delta} = 0, \tag{2.18}$$

$$\{(\alpha - \beta\cos\gamma)\sin\phi_1 + \beta\sin\gamma\cos\phi_1\} f_1 - \frac{2\epsilon N}{mkU}\hat{g} = 0, \tag{2.19}$$

$$\{-(\alpha - \beta\cos\gamma)\cos\phi_1 + \beta\sin\gamma\sin\phi_1\} g_1 + \frac{2\epsilon N}{mkU} f_1 \hat{\delta} = 0, \tag{2.20}$$

$$\{(\alpha - \beta\cos\gamma)\sin\phi_1 + \beta\sin\gamma\cos\phi_1\} g_1 - \frac{2\epsilon N}{mkU}\hat{f} = 0, \tag{2.21}$$

where γ is the phase difference defined in (2.8) between the steady streaming and the variations in stratification. These equations are simply algebraic; together with (2.14) to (2.17) they provide eight equations for the unknown f_1, $\hat{f}$; g_1, $\hat{g}$; δ_1, $\hat{\delta}$ and Δ_1, $\hat{\Delta}$. In view of the purely catalytic role of the steady component of the motion, α and β are constant.

3. The motion in the trapped layer

These field equations must be supplemented by boundary conditions for the motion. For the sake of definiteness, let us consider the possibility of trapping near the ocean surface, whose level relative to the current and density distribution is as yet arbitrary. For internal gravity waves, the free surface condition can be taken as

$$v = 0 \quad \text{at} \quad y = C, \tag{3.1}$$

since the internal wave frequency is small compared with that of free surface waves of the same wave-number (Phillips 1966). From (2.8), then,

$$f_1 \sin(lx + mC - nt + \delta_1) + g_1 \sin(lx - mC - nt + \Delta_1) = 0,$$

for all x, t. To satisfy this condition, it is necessary that the incident and reflected wave amplitudes be equal and the phases differ by π:

$$f_1 = g_1, \quad 2mC + \phi = \pi, \tag{3.2}$$

since $\phi = \phi_1 = \delta_1 - \Delta_1$.

It is convenient now to specify the parameter ϵ whose inverse describes the order of magnitude of the depth over which the wave amplitude varies. The field equations suggest that

$$\epsilon = \frac{Ulk^2}{4mn} = \frac{Uk^3}{4Nm}, \tag{3.3}$$

the latter equality following since $n = Nl/k$. Further, let

$$\epsilon y = Y.$$

The suffices 1 in the field equations can now be dropped and the quantities $\hat{f}$, $\hat{g}$, $\hat{\delta}$ and $\hat{\Delta}$ eliminated from (2.14), (2.20), (2.15) and (2.18). There follows

$$\frac{df}{dY} + \frac{\partial H(\phi)}{\partial \phi} g = 0, \quad \frac{\partial g}{\partial Y} + \frac{\partial H(\phi)}{\partial \phi} f = 0, \tag{3.4}$$

where

$$H(\phi) = 2\alpha\left(\frac{l^2 - m^2}{k^2}\right)\sin\phi - \beta \sin(\phi - \gamma). \tag{3.5}$$

This function can be expressed alternatively as

$$H(\phi) = A \sin(\phi + \phi_0), \tag{3.6}$$

where

$$\left.\begin{aligned} A &= \left[\left\{2\alpha\left(\frac{l^2 - m^2}{k^2}\right) - \beta\cos\gamma\right\}^2 + \beta^2 \sin^2\gamma\right]^{\frac{1}{2}}, \\ \tan\phi_0 &= \frac{\beta \sin\gamma}{[2\alpha(l^2 - m^2)/k^2] - \beta\cos\gamma}, \end{aligned}\right\} \tag{3.7}$$

and without loss of generality, $A > 0$.

A first integral of the set (3.4) follows immediately.

$$ff' - gg' = 0,$$

$$f^2(Y) - g^2(Y) = \text{const.} \tag{3.8}$$

This can be interpreted generally as expressing the constancy of the energy flux in the vertical direction: the vertical components of the group velocities of the two inclined waves being equal in magnitude but opposite in direction and the energy density being proportional to f^2 and g^2. Since there is no net energy flux across the free surface (or equivalently, from the surface condition (3.2)), the constant vanishes and

$$f(Y) = g(Y) \tag{3.9}$$

throughout the region.

The variation in the phase difference ϕ is specified by the remaining pairs of equations (2.16), (2.21) and (2.17), (2.19):

$$\left.\begin{aligned} f\frac{d\delta}{dY} - H(\phi)\,g = 0,\\ g\frac{d\Delta}{dY} + H(\phi)\,f = 0.\end{aligned}\right\} \tag{3.10}$$

Since $\phi = \delta - \Delta$ and $f(Y) = g(Y)$, it follows that

$$\frac{d\phi}{dY} - 2H(\phi) = 0. \tag{3.11}$$

From this equation, it is evident that if $H(\phi) = 0$ at any finite depth Y, then ϕ is constant in the neighbourhood and so everywhere. The only possibilities then are that $H(\phi) = 0$ throughout the region or that $H(\phi)$ has no zeros in any finite interval. If we suppose for the moment that the latter situation should obtain, equation (3.11) can be integrated in the form

$$\begin{aligned} 2A(Y - \epsilon C) &= \int_{\pi - 2mC}^{\phi} \frac{d\phi}{\sin(\phi + \phi_0)} \\ &= \log\left\{\frac{\tan\frac{1}{2}(\phi + \phi_0)}{\tan\frac{1}{2}(\pi - 2mC + \phi_0)}\right\}, \end{aligned} \tag{3.12}$$

since from (3.2), $\phi = \pi - 2mC$ when $Y = \epsilon C$. Now, whatever the sign of the denominator of the argument of the logarithm, as $Y \to -\infty$, the angle ϕ migrates to the nearest zero of the numerator; that is

$$\phi + \phi_0 = 0, \quad 2\pi, \ldots,$$

and $H(\phi) \to 0$. Moreover

$$\frac{dH(\phi)}{d\phi} \to A > 0,$$

and, in virtue of (3.9), the solutions to (3.4) diverge exponentially as $Y \to -\infty$. Consequently, we are forced to conclude that $d\phi/dY = 0$ and $H(\phi) = 0$ everywhere, the appropriate roots now being

$$\phi + \phi_0 = \pi, \ldots.$$

Then

$$\partial H(\phi)/\partial\phi = -A$$

and

$$f(Y) = g(Y) = \text{const.}\, e^{AY}, \tag{3.13}$$

where A is given by (3.7). The attentuation depth L is $(\epsilon A)^{-1}$, or

$$L = \frac{4Nm}{Uk}\{[2\alpha(l^2-m^2)-\beta k^2\cos\gamma]^2+\beta^2k^4\sin^2\gamma\}^{-\frac{1}{2}}. \tag{3.14}$$

The remaining free surface condition (3.2) can be satisfied by appropriate choice of the (as yet) arbitrary level of the free surface

$$C = \phi_0/2m.$$

These solutions describe, as we anticipated, a trapping of the internal waves by their interaction with (or multiple scattering by) the variations in horizontal current or density stratification. The amplitudes of the two internal wave components decrease exponentially with depth

$$f = g \propto \exp(y/L), \tag{3.15}$$

and the energy density is proportional to $\exp(2y/L)$.

Several special cases of interest can be drawn from these results. If, as considered by Brooke Benjamin, the mean density gradient is uniform and the steady current distribution has the maximum value $U\cos\phi$ in the x-direction, then $\beta = 0$ and $\alpha = \cos\phi$. Thus, from (3.14),

$$L = \frac{2Nm}{Uk\cos\phi|l^2-m^2|}.$$

This can be expressed in terms of $2m$, the vertical wave-number of the current field and the angle of inclination θ of the waves to the horizontal. Since

$$k = m/\sin\theta, \quad l = m\cot\theta,$$

$$\begin{aligned} L &= \frac{2N}{m^2U\cos\phi}\left|\frac{\sin^3\theta}{\cos^2\theta-\sin^2\theta}\right| \\ &= \frac{8N}{(2m)^2U\cos\phi}\left|\frac{\sin^3\theta}{\cos 2\theta}\right|, \end{aligned} \tag{3.16}$$

precisely as given by Brooke Benjamin. As he pointed out, a notable property of this solution is the existence of a 'window' for the transmission of waves when $\theta = \frac{1}{4}\pi$ and $n = N/\surd 2$; at this particular frequency the coupling between the variations in current and the waves vanishes and the waves can propagate without interaction. At all other frequencies with the appropriate wavelengths, the depth of the trapped layer is finite and the energy is restricted by multiple scattering to a region of depth $2L$.

Another case of interest is found when the steady horizontal velocity field vanishes (so that $\alpha = 0$) but there are periodic variations in the basic density gradient. If the mean buoyancy gradient is given by

$$-\frac{g}{\rho_0}\frac{\partial\bar{\rho}}{\partial y} = N^2\{1+r\sin(2my+\gamma)\}, \tag{3.17}$$

then, by comparison with (3.8), it follows that

$$\beta = \frac{N}{Uk}r$$

and the penetration depth

$$L = \frac{4Nm}{\beta U k^3} = \frac{4m}{k^2 r},$$
$$= \frac{8}{(2m)r}\sin^2\theta, \tag{3.18}$$

in terms of the vertical wave-number $(2m)$ of the density striations. Clearly, to guarantee static stability of the distribution (3.17) it is necessary that $r < 1$; for the two scale analysis to be valid with $2mL \gg 1$, the stronger condition $r \ll 1$ is required. In this case, there is no 'window', the maximum depth of the trapped layer being found when $\theta \to \frac{1}{2}\pi$ and $n \to 0$.

Again, when the steady horizontal velocity and density variation fields are related as in the limit of an internal gravity wave of zero-frequency

$$\psi = \frac{U}{2m}\cos 2my,$$
$$b = -NU\cos 2my,$$

it follows by comparison with (2.8) that $\alpha = 1, \gamma = 0, \beta = 2m/k$. The penetration depth is then given by

$$L = \frac{2Nm}{Uk}|l^2 - m^2 + mk|^{-1}$$
$$= \frac{2Nm}{Uk^3(1-\sin\theta)(1+2\sin\theta)},$$

where, again, θ is the inclination of the wave-number $\mathbf{k}$ to the horizontal and $\sin\theta = m/k$. Thus

$$L = \frac{8N}{(2m)^2 U}\frac{\sin^3\theta}{(1-\sin\theta)(1+2\sin\theta)}, \tag{3.19}$$

which, again, is finite when $0 < \theta < \frac{1}{2}\pi$. This result differs in detail from the one given earlier (Phillips 1968) because of an algebraic slip there in the calculation of the coupling coefficient, but again there is no 'window'. In fact, the general result (3.14) shows that the situation noted by Brooke Benjamin in which $\beta = 0$ is the only one where this 'window' appears.

Finally, it might be noted that if both α and β vanish, then $L \to \infty$. Two internal waves of the same frequency are mutually transparent; they pass through one another without interaction in a uniformly stratified fluid at rest.

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